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Classical and quantum integrable systems with boundaries

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Abstract. We study two-dimensional classically integrable field theory with independent boundary conditions at each end, and obtain three possible generating functions for integrals of motion when this model is an ultralocal one. Classically integrable boundary conditions can be found when solving boundary $K_{\pm}$ equations. In the quantum case, we also find that the unitarity condition of the quantum R-matrix is sufficient to construct commutative quantities with boundaries, and its reflection equations are obtained.

1. Introduction

Recently, there has been great progress in understanding two-dimensional integrable field theory on a finite interval with independent boundary conditions at each end [1–5]. The motivation is not only from the necessity in itself, but also from studies of boundary-related phenomena in statistical systems near criticality [6] and integrable deformations of conformal field theories [7].

In order to deal with integrable models with boundary, relying on previous results of Cherednik [1], Sklyanin [2, 3] introduced a new generating function which originates from the periodic boundary functions. In classically integrable models [2], if one has the well known relation for the monodramy matrix [8], $\{T \overset{\otimes}{,} T\} = [r,\ T \otimes T]$, and the r-matrix satisfies the condition of $r(\alpha) = -r(-\alpha)$, the new generating function defined in $[x_-, x_+]$ can be expressed as

$$\tau(\alpha) \equiv \mathrm{tr}\{K_+(\alpha)T(x_+, x_-, t, \alpha)K_-(\alpha)T^{-1}(x_+, x_-, t, -\alpha)\} \tag{1}$$

where $K_{\pm}$ are boundary reflection matrices.

Expanded as a Laurent series in α, all coefficients of $\tau(\alpha)$ make an infinite number of integrals of motion which ensure the complete integrability of the model. From [8], $\tau(\alpha)$ must be in involution between different spectral parameter α and it is independent of time. In other words, $K_{\pm}$ must satisfy some constraint equations, and the existence of non-trivial $K_{\pm}$ solutions means there are non-trivial classically integrable boundary conditions (CIBCs).

There is no condition $r(\alpha) = -r(-\alpha)$ in affine Toda field theory (ATFT), so Bowcock *et al* [5] developed a method of a modified Lax pair to deal with such models, in which the new generating function in $(-\infty, x_+]$ reads

$$\tau(\alpha) \equiv \mathrm{tr}\{T^{\dagger}(-\infty, x_+, t, -\alpha)K_+(\alpha)T(-\infty, x_+, t, \alpha)\} \tag{2}$$

in which '†' denotes conjugation and has little difference with the original paper [5] because of the different definition of the T-matrix. We must point out that the boundary Lax pair in [5] has been modified from the periodic boundary Lax pair.

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In this paper, we find it is necessary to add a new parameter to the generating function (1) in order to deal with ATFT, and no symmetry condition of the r-matrix is in fact needed. Besides this modified form, we also construct two other possible generating functions by the zero curvature representation. After we extend our results to quantum integrable systems, we find that the unitarity condition of quantum R-matrix is sufficient to also construct commutative quantities with boundary.

The paper is organized as follows. In section 2, three possible generating functions are constructed by the zero curvature representation. In order to regard the constructed functions as generating functions, algebra equations (reflection equations) and evolution equations of the $K_\pm$ matrices appear in section 3. Then, we study ATFT in section 4 and find the links between these generating functions. In section 5, we extend our results to the quantum case, and demonstrate that the unitarity condition of the quantum R-matrix is sufficient to construct commutative quantities. Then, we compare our commutative quantities with those of paper [13] and find the relation between them. Finally, a discussion will be found in section 6.

2. Construction of generating functions

2.1. Periodic boundary condition

The zero curvature approach to inverse scattering [8] relies on the existence of a pair of linear partial differential equations in a $d \times d$ matrix

$$\partial_x \Psi = U(x, t, \alpha)\Psi \qquad \partial_t \Psi = V(x, t, \alpha)\Psi$$

where the Lax pair (U, V) are $d \times d$ matrices whose elements are functions of the complex-valued field $\phi(x, t)$ and its derivatives, and $\alpha \in C$ is a spectral parameter. The zero curvature condition appears from the compatibility of the above equation:

$$\partial_t U - \partial_x V + [U,\ V] = 0. \tag{3}$$

By the zero curvature representation, we define the transition matrix

$$T(x, y, t, \alpha) = \mathcal{P}\exp\left\{\int_y^x U(x', t, \alpha)\,\mathrm{d}x'\right\} \qquad x \geqslant y \tag{4}$$

where $\mathcal{P}$ denotes a path ordering of non-commuting factors. Now, the T-matrix satisfies

$$\begin{aligned} \partial_x T &= U(x, t, \alpha)T \\ \partial_t T &= V(x, t, \alpha)T - TV(y, t, \alpha) \\ \mathrm{Id} &= T(x, x, t, \alpha), \end{aligned} \tag{5}$$

where Id is the $d \times d$ identity matrix.

It is well known that the trace of the monodramy matrix $T_L(t, \alpha) \equiv T(L, -L, t, \alpha)$ is a generating function with a periodic boundary condition, so we have another more explicit form $\tau(\alpha) = \ln \mathrm{tr} T_L(t, \alpha)$. Expanded as a Laurent series in α, $\tau(\alpha)$ makes an infinite number of integrals of motion. The conservation condition of these integrals can be proved by the second equation of (5) with a periodic boundary condition, and the involution condition is proved in the Poisson bracket

$$\{T(x, y, \alpha) \overset{\otimes}{,} T(x, y, \beta)\} = [r(\alpha, \beta), T(x, y, \alpha) \otimes T(x, y, \beta)] \qquad L \geqslant x \geqslant y \geqslant -L \tag{6}$$

in which T is a $d \times d$ matrix, $\overset{1}{T} \equiv T \otimes \mathrm{Id}$ and $\overset{2}{T} \equiv \mathrm{Id} \otimes T$. $r(\alpha, \beta)$ is a $d^2 \times d^2$ matrix whose elements depend on α and β only. The Jacobi identity for the bracket holds if and only if the r-matrix is a solution of the classical Yang–Baxter equation.

Under the periodic boundary condition, it is obvious that $\{\tau(\alpha) \overset{\otimes}{,} \tau(\beta)\} = 0$. So $\tau(\alpha)$ constructs a family of generating functions for the integrals of motion.

2.2. Independent boundary condition

As soon as the periodic boundary condition is broken, $\tau(\alpha)$ defined before should be not a conservative quantity, so that we have to find a new expression for the generating function. As discussed in [4, 5], if the Lagrangian density in bulk theory is $\mathcal{L}_f$, then the new Lagrangian density with boundary appears as

$$\mathcal{L} = \theta(x_+ - x)\theta(x - x_-)\mathcal{L}_f - \delta(x_+ - x)V_+(\phi(x_+), \partial_\mu\phi(x_+)) \\ -\delta(x - x_-)V_-(\phi(x_-), \partial_\mu\phi(x_-)). \tag{7}$$

By means of the principle of the least action associated with (7), we will obtain the motion equation in (x_-, x_+) and the boundary equations at each end.

If $\mathcal{L}_f$ is expressed as $\mathcal{L}_f = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$ and $V_\pm$ depend only on $\phi(x_\pm)$ (and are independent of the derivatives), the boundary equations at each end will be

$$\partial_x\phi = \mp\partial_\phi V_\pm \qquad x = x_\pm. \tag{8}$$

At each end, if $\phi(x)$ is a smooth function of the coordinate x, we can extend the motion equation to the whole domain $[x_-, x_+]$, so we can do the Lax pair. In other words, we keep the uniform expression of the fundamental Poisson bracket in the whole domain, even if it is an independent boundary condition.

Similar to the definition of the transition matrix, there is another matrix function

$$F(x, t, \alpha) = \mathcal{P}\ \exp\left\{\int_{t_0}^{t} V(x, t', \alpha)\,\mathrm{d}t'\right\} \qquad t \geqslant t_0. \tag{9}$$

By the zero curvature condition, we construct a quantity which is independent of time:

$$F^{-1}(x_+, t_1, \alpha)T(x_+, x_-, t_1, \alpha)F(x_-, t_1, \alpha) = F^{-1}(x_+, t_2, \alpha)T(x_+, x_-, t_2, \alpha)F(x_-, t_2, \alpha). \tag{10}$$

It can be proved easily because both sides in the equation are equal to $T(x_+, x_-, t_0, \alpha)$.

With another equation of argument $(-\alpha + \delta)$, which can be obtained by the same method, a generating function with boundary can be constructed as follows. For example,

$$\mathrm{tr}\{[F(x_+, t, -\alpha + \delta)F^{-1}(x_+, t, \alpha)]\,T(x_+, x_-, t, \alpha) \\ \times[F(x_-, t, \alpha)F^{-1}(x_-, t, -\alpha + \delta)]\,T^{-1}(x_+, x_-, t, -\alpha + \delta)\}.$$

By the product method, it is obvious that this quantity is a conservative quantity. If we regard it as a generating function for integrals of motion, there are two main problems: (1) Does such a constructed quantity satisfy the involution condition? (2) The quantity, which comes from the F-matrix, must be independent of t. Based on these problems, we introduce the $K_\pm$ matrices instead of the F terms and impose involution and conservation conditions on the new form. The new generating function is

(a)

$$\tau(\alpha) = \mathrm{tr}\{K_+(x_+, t, \alpha)T(x_+, x_-, t, \alpha)K_-(x_-, t, \alpha)T^{-1}(x_+, x_-, t, -\alpha + \delta)\}. \tag{11}$$

Moreover, we use '†' (conjugation) or 't' (transposition) instead of '−1' (inverse) in order that K_+ and K_- depend only on the variables of the boundary x_+ and x_-, respectively. The results are:

(b)

$$\tau(\alpha) = \mathrm{tr}\{K_+(x_+, t, \alpha)T(x_+, x_-, t, \alpha)K_-(x_-, t, \alpha)T^{\mathrm{t}}(x_+, x_-, t, -\alpha + \delta)\} \tag{12}$$

(c)

$$\tau(\alpha) = \mathrm{tr}\{K_+(x_+, t, \alpha)T(x_+, x_-, t, \alpha)K_-(x_-, t, \alpha)T^{\dagger}(x_+, x_-, t, -\alpha + \delta)\}. \tag{13}$$

In (11)–(13), the $K_\pm$ matrices and δ are similar in symbols only. Each of these quantities will be a generating function of integrable systems with boundary, if both the involution and conservation conditions are satisfied.

3. $\tau(\alpha)$ as a generating function

It is well known that generating functions for integrals of motion must be in involution (between each other) and independent of time [8]. So if we regard quantity (11) as a generating function, some constraint conditions must be imposed on it. Now we study quantity (11) in this section.

3.1. Involution condition

Taking notation similar to paper [3], we define

$$\begin{aligned} \mathcal{T}_+(x,\alpha) &= T^{-1}(x_+,x,t,-\alpha+\delta)K_+(x_+,t,\alpha)T(x_+,x,t,\alpha)\\ \mathcal{T}_-(x,\alpha) &= T(x,x_-,t,\alpha)K_-(x_-,t,\alpha)T^{-1}(x,x_-,t,-\alpha+\delta)\\ \mathcal{T}(x,\alpha) &= \mathcal{T}_-(x,\alpha)\mathcal{T}_+(x,\alpha). \end{aligned}$$

In comparison with quantity (11), we find $\mathrm{tr}\mathcal{T}(x,\alpha)$ is just equal to $\tau(\alpha)$. Now, we impose some constraint conditions on the $K_\pm$ matrices:

$$\begin{aligned} &\{K_\pm(x_\pm,t,\alpha) \overset{\otimes}{,} T(x_+,x_-,t,\beta)\} = 0\\ &\{K_\pm(\alpha) \overset{\otimes}{,} K_\pm(\beta)\} = 0 \qquad \{K_\pm(\alpha) \overset{\otimes}{,} K_\mp(\beta)\} = 0. \end{aligned} \tag{14}$$

This implies

$$\{\mathcal{T}_+(x,t) \overset{\otimes}{,} \mathcal{T}_-(x,t)\} = 0.$$

If the $K_\pm$ matrices are independent of field variances, condition (14) is satisfied naturally. However, we must point out that the $K_\pm$ matrices, in general, can depend on the field variables. The Poisson bracket on $\mathcal{T}_\pm$ is as follows.

Proposition 1. *If the K_+ matrix satisfies*

$$\begin{aligned} 0 = &-r(-\alpha+\delta,-\gamma+\delta)\overset{1}{K}_+(t,\alpha)\overset{2}{K}_+(t,\gamma)+\overset{1}{K}_+(t,\alpha)r(\alpha,-\gamma+\delta)\overset{2}{K}_+(t,\gamma)\\ &+\overset{2}{K}_+(t,\gamma)r(-\alpha+\delta,\gamma)\overset{1}{K}_+(t,\alpha)-\overset{1}{K}_+(t,\alpha)\overset{2}{K}_+(t,\gamma)r(\alpha,\gamma) \end{aligned} \tag{15}$$

then the $\mathcal{T}_+$ algebra should obey the following relation:

$$\begin{aligned} \{\overset{1}{\mathcal{T}}_+(x,\alpha),\overset{2}{\mathcal{T}}_+(x,\gamma)\} = &-r(-\alpha+\delta,-\gamma+\delta)\overset{1}{\mathcal{T}}_+(x,\alpha)\overset{2}{\mathcal{T}}_+(x,\gamma)\\ &+\overset{1}{\mathcal{T}}_+(x,\alpha)r(\alpha,-\gamma+\delta)\overset{2}{\mathcal{T}}_+(x,\gamma)+\overset{2}{\mathcal{T}}_+(x,\gamma)r(-\alpha+\delta,\gamma)\overset{1}{\mathcal{T}}_+(x,\alpha)\\ &-\overset{1}{\mathcal{T}}_+(x,\alpha)\overset{2}{\mathcal{T}}_+(x,\gamma)r(\alpha,\gamma). \end{aligned} \tag{16}$$

It should be emphasized that K_+ is a subalgebra of the $\mathcal{T}_+$ algebra according to the definition of $\mathcal{T}_+$. Proposition 1 can be proved by calculating the Poisson bracket on $\mathcal{T}_+$ directly. There is another algebra of $\mathcal{T}_-$ similar to $\mathcal{T}_+$.

Proposition 2. *If the K_- matrix satisfies*

$$\begin{aligned} 0 = &\,r(\alpha,\gamma)\overset{1}{K}_-(t,\alpha)\overset{2}{K}_-(t,\gamma)-\overset{1}{K}_-(t,\alpha)r(-\alpha+\delta,\gamma)\overset{2}{K}_-(t,\gamma)\\ &-\overset{2}{K}_-(t,\gamma)r(\alpha,-\gamma+\delta)\overset{1}{K}_-(t,\alpha)\\ &+\overset{1}{K}_-(t,\alpha)\overset{2}{K}_-(t,\gamma)r(-\alpha+\delta,-\gamma+\delta) \end{aligned} \tag{17}$$

then it leads to the relation of the $\mathcal{T}_-$ algebra being

$$\begin{aligned}
\{\overset{1}{\mathcal{T}}_-(x,\alpha),\overset{2}{\mathcal{T}}_-(x,\gamma)\} = r(\alpha,\gamma)\overset{1}{\mathcal{T}}_-(x,\alpha)\overset{2}{\mathcal{T}}_-(x,\gamma) &- \overset{1}{\mathcal{T}}_-(x,\alpha)r(-\alpha+\delta,\gamma)\overset{2}{\mathcal{T}}_-(x,\gamma)\\
&- \overset{2}{\mathcal{T}}_-(x,\gamma)r(\alpha,-\gamma+\delta)\overset{1}{\mathcal{T}}_-(x,\alpha)\\
&+ \overset{1}{\mathcal{T}}_-(x,\alpha)\overset{2}{\mathcal{T}}_-(x,\gamma)r(-\alpha+\delta,-\gamma+\delta).
\end{aligned}\tag{18}$$

The proof is similar to that of proposition 1. Used propositions 1 and 2, the Poisson bracket on $\mathcal{T}(x,\alpha)$ can be calculated as follows:

$$\begin{aligned}
\{\overset{1}{\mathcal{T}}(x,\alpha),\overset{2}{\mathcal{T}}(x,\gamma)\} &= \{\overset{1}{\mathcal{T}}_-(x,\alpha)\overset{1}{\mathcal{T}}_+(x,\alpha),\overset{2}{\mathcal{T}}_-(x,\gamma)\overset{2}{\mathcal{T}}_+(x,\gamma)\}\\
&= \overset{1}{\mathcal{T}}_-(x,\alpha)\overset{2}{\mathcal{T}}_-(x,\gamma)\{\overset{1}{\mathcal{T}}_+(x,\alpha),\overset{2}{\mathcal{T}}_+(x,\gamma)\}\\
&\quad+\{\overset{1}{\mathcal{T}}_-(x,\alpha),\overset{2}{\mathcal{T}}_-(x,\gamma)\}\overset{1}{\mathcal{T}}_+(x,\alpha)\overset{2}{\mathcal{T}}_+(x,\gamma)\\
&= [r(\alpha,\gamma),\overset{1}{\mathcal{T}}(x,\alpha)\overset{2}{\mathcal{T}}(x,\gamma)] + [\overset{1}{\mathcal{T}}(x,\alpha),\overset{2}{\mathcal{T}}_-(x,\gamma)r(\alpha,-\gamma+\delta)\overset{2}{\mathcal{T}}_+(x,\gamma)]\\
&\quad+[\overset{2}{\mathcal{T}}(x,\gamma),\overset{1}{\mathcal{T}}_-(x,\alpha)r(-\alpha+\delta,\gamma)\overset{1}{\mathcal{T}}_+(x,\alpha)].
\end{aligned}$$

After taking the trace of $\mathcal{T}$, we find

$$\{\mathrm{tr}\,\overset{1}{\mathcal{T}}(x,\alpha),\mathrm{tr}\,\overset{2}{\mathcal{T}}(x,\gamma)\} = \mathrm{tr}_1\mathrm{tr}_2\{\overset{1}{\mathcal{T}}(x,\alpha),\overset{2}{\mathcal{T}}(x,\gamma)\} = 0$$

that is,

$$\{\overset{1}{\tau}(\alpha),\overset{2}{\tau}(\gamma)\} = 0. \tag{19}$$

In other words, $\tau(\alpha)$ constructs a one-parameter involutive family. Here we note that no symmetry conditions of the r-matrix are used to obtain equation (19). Consequently, it can be applied to the general model.

3.2. Conservation condition

If $\tau(\alpha)$ is a generating function for integrals of motion, it must be independent of time. We find

$$\begin{aligned}
\partial_t\mathrm{tr}\mathcal{T}(x,\alpha) &= \partial_t\mathrm{tr}\{K_+(t,\alpha)T(x_+,x_-,t,\alpha)K_-(t,\alpha)T^{-1}(x_+,x_-,t,-\alpha+\delta)\}\\
&= \mathrm{tr}\{[\partial_t K_+(t,\alpha) - V(x_+,t,-\alpha+\delta)K_+(t,\alpha) + K_+(t,\alpha)V(x_+,t,\alpha)]\\
&\quad\times T(x_+,x_-,t,\alpha)K_-(t,\alpha)T^{-1}(x_+,x_-,t,-\alpha+\delta)\\
&\quad+[\partial_t K_-(t,\alpha) - V(x_-,t,\alpha)K_-(t,\alpha) + K_-(t,\alpha)V(x_-,t,-\alpha+\delta)]\\
&\quad\times T^{-1}(x_+,x_-,t,-\alpha+\delta)K_+(t,\alpha)T(x_+,x_-,t,\alpha)\}.
\end{aligned}\tag{20}$$

Taking $\partial_t\mathrm{tr}\mathcal{T}(x,\alpha) = 0$, and supposing there is no connection between the boundary variances at each end, we obtain the evolution equations of the $K_\pm$ matrices

$$\begin{aligned}
&\partial_t K_+(t,\alpha) - V(x_+,t,-\alpha+\delta)K_+(t,\alpha) + K_+(t,\alpha)V(x_+,t,\alpha) = 0\\
&\partial_t K_-(t,\alpha) - V(x_-,t,\alpha)K_-(t,\alpha) + K_-(t,\alpha)V(x_-,t,-\alpha+\delta) = 0.
\end{aligned}\tag{21}$$

For these equations, we find immediately that there are two isomorphisms between K_+ and K_-, which are $K_+(\alpha)\to K_-(-\alpha+\delta)$ and $K_+(\alpha)\to K_-^{-1}(\alpha)$.

If $K_\pm$ are constant matrices ($\partial_t K_\pm = 0$), equation (21) can be simplified as

$$\begin{aligned}
&V(x_+,t,-\alpha+\delta)K_+(t,\alpha) = K_+(t,\alpha)V(x_+,t,\alpha)\\
&V(x_-,t,\alpha)K_-(t,\alpha) = K_-(t,\alpha)V(x_-,t,-\alpha+\delta).
\end{aligned}\tag{22}$$

In the present case, we note that the $K_\pm$ matrices are not singular matrices, so that the determinants of V satisfy

$$\det V(x_\pm, t, -\alpha + \delta) = \det V(x_\pm, t, \alpha) \tag{23}$$

by which we can obtain the value of δ. After inserting the δ value into equation (22), we can find some non-trivial CIBCs when non-trivial $K_\pm$ matrices appear. In other words, a class of $K_\pm$ matrices is related to a class of integrable boundary conditions.

We must point out that $K_\pm$ matrices depending on field variables have their meaning in fact [9]. On the one hand, we must use such $K_\pm$ matrices in order that quantity (11) can also be regarded as a generating function on the periodic boundary condition. On the other hand, studying such $K_\pm$ matrices, we understand the integrable condition more deeply.

When Sklyanin's function (1) is regarded as a generating function in ATFT, boundary $K_\pm$ matrices will have no constant solution in equation (21) (except for sine–Gordon theory). So we have to solve equation (21) as differential equations. Apart from this difficulty, even when one has found a non-trivial solution, one had to prove the involution condition again because condition (14) may be broken. In our method, δ added to spectral parameter guarantees the existence of constant $K_\pm$ matrices, and δ can be solved by means of equation (23), so the $K_\pm$ matrices solving procedure is simplified effectively.

Proposition 3. *If $K_\pm$ matrices satisfy not only the algebra equations (15) and (17), but also the evolution equations of (21),* $\mathrm{tr}\mathcal{T}(x, \alpha)$ *is a generating function for the integrals of motion.*

There is a difficult step in proving this proposition. Are solutions of the $K_\pm$ matrices in equation (21) compatible with the algebra equations (15) and (17)? Although this is true in sine–Gordon theory [10] and it has been proved in ATFT [5] with the form of (12), it is still an open problem in general theory. If this compatibility is satisfied, the proposition is proved naturally.

3.3. *Other generating functions*

As exhibited in the above subsections, we also obtain the algebra and evolution equations of other $K_\pm$ matrices when quantities (12) or (13) are regarded as a generating function for the integrals of motion. In the form of (12), we have

$$\begin{aligned}
0 = {} & r^{t_1t_2}(-\alpha+\delta, -\gamma+\delta) \overset{1}{K}_+(t,\alpha) \overset{2}{K}_+(t,\gamma) + \overset{1}{K}_+(t,\alpha) r^{t_2}(\alpha, -\gamma+\delta) \overset{2}{K}_+(t,\gamma) \\
& + \overset{2}{K}_+(t,\gamma) r^{t_1}(-\alpha+\delta, \gamma) \overset{1}{K}_+(t,\alpha) + \overset{1}{K}_+(t,\alpha) \overset{2}{K}_+(t,\gamma) r(\alpha,\gamma) \\
0 = {} & r(\alpha,\gamma) \overset{1}{K}_-(t,\alpha) \overset{2}{K}_-(t,\gamma) + \overset{1}{K}_-(t,\alpha) r^{t_1}(-\alpha+\delta,\gamma) \overset{2}{K}_-(t,\gamma) \\
& + \overset{2}{K}_-(t,\gamma) r^{t_2}(\alpha, -\gamma+\delta) \overset{1}{K}_-(t,\alpha) \\
& + \overset{1}{K}_-(t,\alpha) \overset{2}{K}_-(t,\gamma) r^{t_1t_2}(-\alpha+\delta, -\gamma+\delta)
\end{aligned} \tag{24}$$

where the upper indices '$t_i, i = 1, 2$' denote transposition on the 'i' space. The evolution equations are

$$\begin{aligned}
0 &= \partial_t K_+(t,\alpha) + V^t(x_+, t, -\alpha+\delta) K_+(t,\alpha) + K_+(t,\alpha) V(x_+, t, \alpha) \\
0 &= \partial_t K_-(t,\alpha) - V(x_-, t, \alpha) K_-(t,\alpha) - K_-(t,\alpha) V^t(x_-, t, -\alpha+\delta).
\end{aligned} \tag{25}$$

Let $\partial_t K_\pm = 0$, we can also obtain the δ value in (25) by taking determinants. With the constraint conditions of (24) and (25), quantity (12) is a generating function for the integrals of motion.

In the form of (13), the similar equations are still balanced except that we must use '†' (conjugation) instead of 't' (transposition). If these modified equations are satisfied, quantity (13) can also be regarded as a generating function for the integrals of motion.

Now, we have obtained three forms of generating function as well as their constraint conditions. If they are all regarded as generating functions, we believe in fact that they are the same. In the next section, we will prove it explicitly in ATFT.

4. Classically integrable boundary condition in ATFT

4.1. Links among generating function

The Lagrangian in ATFT with independent boundary condition is [5, 11]

$$\mathcal{L} = \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}t \left\{ \theta(x - x_-)\theta(x_+ - x) \left[\frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - \frac{m^2}{\beta^2} \sum_0^r n_i (\mathrm{e}^{\beta\alpha_i \cdot \phi} - 1) \right] \right.$$
$$\left. - \delta(x - x_-) V_-(\phi(x_-), \partial_\mu \phi(x_-)) - \delta(x_+ - x) V_+(\phi(x_+), \partial_\mu \phi(x_+)) \right\} \qquad (26)$$

where m is the mass scale and β is the coupling constant in the real domain; α_i are simple roots of a simple Lie algebra of rank r (including the affine root α_0). We have $\sum_0^r n_i \alpha_i = 0$ and $n_0 = 1$. This is a theory of r scalar fields ($\alpha_i \cdot \phi = \sum_{a=0}^{r-1} \alpha_i^a \phi_a$). The potentials V_+ and V_- are additions on the ends x_+ and x_-, respectively. They denote independent boundary conditions. When $V_\pm$ depend on $\phi(x_\pm)$ only (and are independent of the derivatives), we obtain

$$(\partial_t^2 - \partial_x^2)\phi = -\frac{m^2}{\beta} \sum_0^r n_i \alpha_i \, \mathrm{e}^{\beta\alpha_i \cdot \phi} \qquad x_- < x < x_+$$
$$\partial_x \phi_a = \mp \frac{\partial V_\pm}{\partial \phi_a} \qquad x = x_\pm. \qquad (27)$$

The Lax pair in ATFT reads as ($\lambda = \mathrm{e}^\alpha$)

$$U(x, t, \lambda) = -\left\{ \frac{1}{2}\beta H \cdot \partial_t \phi + m \sum_0^r \sqrt{m_i} (\lambda E_{\alpha_i} + \lambda^{-1} E_{-\alpha_i}) \, \mathrm{e}^{\beta\alpha_i \cdot \phi/2} \right\}$$
$$V(x, t, \lambda) = -\left\{ \frac{1}{2}\beta H \cdot \partial_x \phi + m \sum_0^r \sqrt{m_i} (\lambda E_{\alpha_i} - \lambda^{-1} E_{-\alpha_i}) \, \mathrm{e}^{\beta\alpha_i \cdot \phi/2} \right\} \qquad (28)$$

in which H and $E_{\pm\alpha_i}$ are the Cartan subalgebra and the generators responding to the simple roots, respectively, of the simple Lie algebra of rank r. The coefficients m_i are equal to $n_i \alpha_i^2/8$. We have the Lie algebra relation

$$[H_i, H_j] = 0 \qquad [H, E_{\pm\alpha_i}] = \pm\alpha_i E_{\pm\alpha_i}$$
$$[E_{\alpha_i}, E_{-\alpha_i}] = 2\alpha_i \cdot H/(\alpha_i^2). \qquad (29)$$

It has been pointed out by Hollwood [12] that the complex affine Toda theories have soliton solutions (in which the coupling constant β is purely imaginary), in contrast with the real coupling constant. The Lagrangian, motion equation and Lax pair in [12] can be expressed similarly to equations (26)–(28) except for taking $\beta \to \mathrm{i}\tilde{\beta}$ ($\tilde{\beta} \in Re$). In our paper, we use equations (26)–(28) in general, and distinguish them only when the real and imaginary cases can not be treated in the same way.

For those generators in the Lax pair (28), we can find a representation in which they satisfy

$$H_i^t = H_i^\dagger = H_i \qquad E_{\pm\alpha_i}^{\mathrm{t}} = E_{\pm\alpha_i}^\dagger = E_{\mp\alpha_i}.$$

So there is an automorphism map:

$$\begin{aligned}
&H_i \to H_i' = \Omega^{-1} H_i \Omega = -H_i \\
&E_{\alpha_i} \to E_{\alpha_i}' = \Omega^{-1} E_{\alpha_i} \Omega = E_{-\alpha_i} \\
&E_{-\alpha_i} \to E_{-\alpha_i}' = \Omega^{-1} E_{-\alpha_i} \Omega = E_{\alpha_i}.
\end{aligned}$$

The new generators satisfy the same Lie algebra relation (29). In other words, we have

$$\begin{aligned}
U^{\mathrm{t}}(x,\lambda) &= U(x,\lambda^{-1}) = -\Omega^{-1} U(x,-\lambda)\Omega \\
V^{\mathrm{t}}(x,\lambda) &= V(x,-\lambda^{-1}) = -\Omega^{-1} V(x,-\lambda)\Omega.
\end{aligned} \tag{30}$$

We note that equation (30) can be applied to both real and imaginary coupling constant cases. However, if one uses '†' instead of 't', equation (30) must be modified because of its complex fields.

From the definition of $T(x, y, t, \lambda)$, we have

$$\begin{aligned}
\partial_x T^{\mathrm{t}}(x,y,\lambda) &= T^{\mathrm{t}}(x,y,\lambda) U^{\mathrm{t}}(x,\lambda) \\
&= T^{\mathrm{t}}(x,y,\lambda)[-\Omega^{-1} U(x,-\lambda)\Omega]
\end{aligned}$$

or

$$\partial_x[\Omega T^{\mathrm{t}}(x,y,\lambda)\Omega^{-1}] = -[\Omega T^{\mathrm{t}}(x,y,\lambda)\Omega^{-1}] U(x,-\lambda).$$

Comparing this with $\partial_x T^{-1}(x,y,\lambda) = -T^{-1}(x,y,\lambda)U(x,\lambda)$ and the initial condition in (5), we obtain

$$\Omega T^{\mathrm{t}}(x,y,\lambda)\Omega^{-1} = T^{-1}(x,y,-\lambda)$$

or

$$\Omega T^{\mathrm{t}}(x,y,\alpha)\Omega^{-1} = T^{-1}(x,y,\alpha+\mathrm{i}\pi). \tag{31}$$

This implies

$$\begin{aligned}
\mathrm{tr}\{K_-(\alpha)T^{-1}(-\alpha+\delta)K_+(\alpha)T(\alpha)\} &= \mathrm{tr}\{K_-(\alpha)\Omega T^{\mathrm{t}}(-\alpha+\delta+\mathrm{i}\pi)\Omega^{-1}K_+(\alpha)T(\alpha)\} \\
&= \mathrm{tr}\{\tilde{K}_-(\alpha)T^{\mathrm{t}}(-\alpha+\delta')\tilde{K}_+(\alpha)T(\alpha)\}
\end{aligned} \tag{32}$$

in which the $K_\pm$ matrices in (11) and (12) are now distinguished by $K_\pm$ and $\tilde{K}_\pm$, and the quantities added onto the spectral parameter become δ and δ', respectively. In other words, quantity (11) is equal to (12), if δ' is equal to $\delta + \mathrm{i}\pi$ and the reflection matrices satisfy

$$\tilde{K}_-(\alpha) = K_-(\alpha)\Omega \qquad \tilde{K}_+(\alpha) = \Omega^{-1}K_+(\alpha). \tag{33}$$

Using the second equation of (30) and comparing equation (21) with (25), we find that these relations appear again. So quantities (11) and (12) are in fact the same, when both are regarded as generating functions for the integrals of motion.

In the real coupling constant and real fields case, if we use '†' instead of 't', equation (30) is still balanced when the spectral parameter is real. So we obtain a relation similar to (32) again. In this case, when we rewrite (13) as $\tau(\alpha) = \mathrm{tr}\bar{K}_-(\alpha)T^\dagger(-\alpha+\delta'')\bar{k}_+(\alpha)T(\alpha)$, then

$$\tilde{K}_\pm(\alpha) = \bar{K}_\pm(\alpha) \qquad \delta' = \delta'' = \delta + \mathrm{i}\pi. \tag{34}$$

Now, we have proved quantity (11) is equal to (12) when both are regarded as generating functions in ATFT. When the coupling constant is real, they are also equal to the generating function (13). However, when the coupling constant is purely imaginary, equation (30) may be not satisfied, so the $K_\pm$ matrices in (13) may have no constant solution.

4.2. Classically integrable boundary condition

In real coupling constant ATFT, if we regard $\tau(\alpha) = \mathrm{tr}\bar{K}_-(\alpha)T^\dagger(-\alpha+\delta)\bar{K}_+(\alpha)T(\alpha)$ as a generating function, we will obtain the evolution equation of $\bar{K}_+$ on the x_+ boundary:

$$V^\dagger(x_+, -\alpha+\delta)\bar{K}_+(\alpha) + \bar{K}_+(\alpha)V(x_+, \alpha) = 0. \tag{35}$$

After taking $\delta = 0$, we obtain the equation for $\bar{K}_+^{-1}(\lambda)$ (in which $\lambda = \mathrm{e}^\alpha$):

$$\frac{1}{2}\left[\bar{K}_+^{-1}(\lambda), \frac{\beta}{m}\partial_x\phi\cdot H\right]_+ = \left[\bar{K}_+^{-1}(\lambda), \sum_0^r \sqrt{m_i}(\lambda E_{\alpha_i} - \lambda^{-1}E_{-\alpha_i})\mathrm{e}^{\beta\alpha_i\cdot\phi/2}\right]_-. \tag{36}$$

By the boundary equation (27), this is just the reflection equation that appears in [5]. We find that the K_+ and T matrices defined in [5] are just the quantities of $\bar{K}_+^{-1}$ and T^{-1} in our paper, according to a different definition of the Lax pair. In analogy with the method in [5], we solve equation (36) and obtain CIBCs in ATFT. In the simple-laced case, we have the same as in [5],

$$\frac{\beta}{m}\partial_x\phi = -\sum_0^r B_i\sqrt{\frac{n_i}{2|\alpha_i|^2}}\alpha_i\mathrm{e}^{\beta\alpha_i\cdot\phi/2}$$

in which

$$|B_i| = 2 \qquad i = 0, 1, \ldots, r \qquad \text{or} \qquad B_i = 0 \qquad i = 0, 1, \ldots, r. \tag{37}$$

In the imaginary coupling constant case, we regard $\tau(\alpha) = \mathrm{tr}\tilde{K}_-(\alpha)T^{\mathrm{t}}(-\alpha+\delta)\tilde{K}_+(\alpha)T(\alpha)$ as a generating function. The results in the real coupling constant case can be used thanks to section 4.1. In other words, the new CIBCs can be obtained by analytic continuation by $\beta \to \mathrm{i}\tilde{\beta}$ ($\tilde{\beta} \in Re$). Therefore,

$$\frac{\mathrm{i}\tilde{\beta}}{m}\partial_x\phi = -\sum_0^r B_i\sqrt{\frac{n_i}{2|\alpha_i|^2}}\alpha_i\mathrm{e}^{\mathrm{i}\tilde{\beta}\alpha_i\cdot\phi/2}$$

in which

$$|B_i| = 2 \qquad i = 0, 1, \ldots, r \qquad \text{or} \qquad B_i = 0 \qquad i = 0, 1, \ldots, r. \tag{38}$$

We remark that Sklyanin's method [3] cannot be used in ATFT except for sine–Gorden theory; this conclusion comes from the fact that $\delta \neq 0$ under the independent boundary condition if we regard (11) as a generating function. Now, we must take $\delta = -\mathrm{i}\pi$ according to equations (33) and (34). Sine–Gordon theory is an exception in which it is satisfied both for $\delta = 0$ and $\delta = -\mathrm{i}\pi$.

As we discussed in section 3, in the classical case, no symmetry conditions of the r-matrix are necessary in constructing the generating function for the integrals of motion. So it is interesting to study whether commutative quantities can be constructed with less symmetry of the R-matrix in the quantum case.

5. Quantum integrable systems with boundary

There are many papers (for example, [3, 13–19]) in which the authors deal with quantum integrable boundary conditions in two-dimensional lattice models. As far as we know, both unitarity and crossing unitarity conditions (or the weaker property [16]) are used in constructing commutative quantities. Since finding the crossing unitarity condition of a given R-matrix is a difficult problem, it is useful to construct commutative quantities without this symmetry.

In this section, we explore how to obtain commutative quantities by means of the unitarity condition only. The unitarity condition reads as

$$R_{12}(u)R_{21}(-u)=\xi(u) \tag{39}$$

where $\xi(u)$ is some even scalar function and the R-matrix is a solution of the quantum Yang–Baxter equation (YBE):

$$R_{12}(u-v)R_{13}(u)R_{23}(v)=R_{23}(v)R_{13}(u)R_{12}(u-v).$$

As usual, the transfer matrix $t(u)$ is defined as

$$t(u)=\mathrm{tr}\mathcal{T}_+(u)\mathcal{T}_-(u) \tag{40}$$

and each entry of $\mathcal{T}_+(u)$ commutes with $\mathcal{T}_-(u)$.

Proposition 4. *If $\mathcal{T}_\pm$ satisfy such equations*

$$\begin{aligned}
&R_{21}^{t_1t_2}(-u_-)\overset{1}{\mathcal{T}}{}_+^{t_1}(u)R_{21}^{t_2}(u_+-\delta)\overset{2}{\mathcal{T}}{}_+^{t_2}(v)=\overset{2}{\mathcal{T}}{}_+^{t_2}(v)R_{12}^{t_1}(u_+-\delta)\overset{1}{\mathcal{T}}{}_+^{t_1}(u)R_{12}(-u_-)\\
&R_{12}(u_-)\overset{1}{\mathcal{T}}_-(u)R_{12}^{t_1}(-u_++\delta)\overset{2}{\mathcal{T}}_-(v)=\overset{2}{\mathcal{T}}_-(v)R_{21}^{t_2}(-u_++\delta)\overset{1}{\mathcal{T}}_-(u)R_{21}^{t_1t_2}(u_-)
\end{aligned} \tag{41}$$

and the quantum R-matrix obeys the unitarity condition, then the transfer matrix $t(u)$ defines a one-parameter commutative family.

To be explicit, we use ξ_1^{-1} and ξ_2^{-1} to replace $\xi^{-1}(u_+-\delta)$ and $\xi^{-1}(-u_-)$, respectively, as well as $u_\pm=u\pm v$. The proof is direct:

$$\begin{aligned}
t(u)t(v)&=\mathrm{tr}_1\overset{1}{\mathcal{T}}_+(u)\overset{1}{\mathcal{T}}_-(u)\mathrm{tr}_2\overset{2}{\mathcal{T}}_+(v)\overset{2}{\mathcal{T}}_-(v)=\mathrm{tr}_{12}\overset{1}{\mathcal{T}}{}_+^{t_1}(u)\overset{2}{\mathcal{T}}_+(v)\overset{1}{\mathcal{T}}{}_-^{t_1}(u)\overset{2}{\mathcal{T}}_-(v)\\
&=\xi_1^{-1}\mathrm{tr}_{12}\overset{1}{\mathcal{T}}{}_+^{t_1}\overset{2}{\mathcal{T}}_+R_{21}(u_+-\delta)R_{12}(-u_++\delta)\overset{1}{\mathcal{T}}{}_-^{t_1}\overset{2}{\mathcal{T}}_-\\
&=\xi_1^{-1}\mathrm{tr}_{12}\{\overset{1}{\mathcal{T}}{}_+^{t_1}R_{21}^{t_2}(u_+-\delta)\overset{2}{\mathcal{T}}{}_+^{t_2}\}^{t_2}\{\overset{1}{\mathcal{T}}_-R_{12}^{t_1}(-u_++\delta)\overset{2}{\mathcal{T}}_-\}^{t_1}\\
&=\xi_1^{-1}\xi_2^{-1}\mathrm{tr}_{12}\{\overset{1}{\mathcal{T}}{}_+^{t_1}R_{21}^{t_2}(u_+-\delta)\overset{2}{\mathcal{T}}{}_+^{t_2}\}^{t_1t_2}R_{21}(-u_-)R_{12}(u_-)\{\overset{1}{\mathcal{T}}_-R_{12}^{t_1}(-u_++\delta)\overset{2}{\mathcal{T}}_-\}\\
&=\xi_1^{-1}\xi_2^{-1}\mathrm{tr}_{12}\{R_{21}^{t_1t_2}(-u_-)\overset{1}{\mathcal{T}}{}_+^{t_1}R_{21}^{t_2}(u_+-\delta)\overset{2}{\mathcal{T}}{}_+^{t_2}\}^{t_1t_2}\\
&\quad\times\{R_{12}(u_-)\overset{1}{\mathcal{T}}_-R_{12}^{t_1}(-u_++\delta)\overset{2}{\mathcal{T}}_-\}.
\end{aligned}$$

Using equations (41), we find

$$\begin{aligned}
t(u)t(v)&=\xi_1^{-1}\xi_2^{-1}\mathrm{tr}_{12}\{\overset{2}{\mathcal{T}}{}_+^{t_2}R_{12}^{t_1}(u_+-\delta)\overset{1}{\mathcal{T}}{}_+^{t_1}R_{12}(-u_-)\}^{t_1t_2}\{\overset{2}{\mathcal{T}}_-R_{21}^{t_2}(-u_++\delta)\overset{1}{\mathcal{T}}_-R_{21}^{t_1t_2}(u_-)\}\\
&=\xi_1^{-1}\mathrm{tr}_{12}\{\overset{2}{\mathcal{T}}{}_+^{t_2}R_{12}^{t_1}(u_+-\delta)\overset{1}{\mathcal{T}}{}_+^{t_1}\}^{t_1t_2}\{\overset{2}{\mathcal{T}}_-R_{21}^{t_2}(-u_++\delta)\overset{1}{\mathcal{T}}_-\}\\
&=\xi_1^{-1}\mathrm{tr}_{12}\{\overset{2}{\mathcal{T}}{}_+^{t_2}R_{12}^{t_1}(u_+-\delta)\overset{1}{\mathcal{T}}{}_+^{t_1}\}^{t_1}\{\overset{2}{\mathcal{T}}_-R_{21}^{t_2}(-u_++\delta)\overset{1}{\mathcal{T}}_-\}^{t_2}\\
&=\mathrm{tr}_{12}\overset{2}{\mathcal{T}}{}_+^{t_2}\overset{1}{\mathcal{T}}_+\overset{2}{\mathcal{T}}{}_-^{t_2}\overset{1}{\mathcal{T}}_-\\
&=t(v)t(u).
\end{aligned}$$

In the quantum spin chain model, it is convenient that $\mathcal{T}_\pm$ take such representations as

$$\begin{aligned}
\mathcal{T}_+(u)&=K_+(u)\\
\mathcal{T}_-(u)&=T(u)K_-(u)T^{t}(-u+\delta)\\
&=L_N(u)\ldots L_2(u)L_1(u)K_-(u)L_1^{t}(-u+\delta)L_2^{t}(-u+\delta)\ldots L_N^{t}(-u+\delta)
\end{aligned} \tag{42}$$

in which the transposition 't' acts on the auxiliary space and $n = 1, 2, \ldots, N$ denote quantum space. There is a relation between R and L operators

$$R_{ab}(u-v)L_a(u)L_b(v) = L_b(v)L_a(u)R_{ab}(u-v). \tag{43}$$

Letting $\mathcal{T}_-(u) = L_N(u)\mathcal{T}'_-(u)L_N^{\mathrm{t}}(-u+\delta)$ and inserting equation (42) into (41), we find that the second equation of (41) becomes

$$\begin{aligned}
\mathrm{LHS} &= R_{ab}(u_-)L_{aN}(u) \overset{a}{\mathcal{T}'_-} L^{\mathrm{t}_a}_{aN}(-u+\delta)R^{\mathrm{t}_a}_{ab}(-u_+ +\delta)L_{bN}(v) \overset{b}{\mathcal{T}'_-} L^{\mathrm{t}_b}_{bN}(-v+\delta) \\
&= R_{ab}(u_-)L_{aN}(u) \overset{a}{\mathcal{T}'_-} L_{bN}(v)R^{\mathrm{t}_a}_{ab}(-u_+ +\delta)L^{\mathrm{t}_a}_{aN}(-u+\delta) \overset{b}{\mathcal{T}'_-} L^{\mathrm{t}_b}_{bN}(-v+\delta) \\
&= L_{bN}(v)L_{aN}(u)R_{ab}(u_-) \overset{a}{\mathcal{T}'_-} R^{\mathrm{t}_a}_{ab}(-u_+ +\delta) \overset{b}{\mathcal{T}'_-} L^{\mathrm{t}_a}_{aN}(-u+\delta)L^{\mathrm{t}_b}_{bN}(-v+\delta)
\end{aligned}$$

$$\begin{aligned}
\mathrm{RHS} &= L_{bN}(v) \overset{b}{\mathcal{T}'_-} L^{\mathrm{t}_b}_{bN}(-v+\delta)R^{\mathrm{t}_b}_{ba}(-u_+ +\delta)L_{aN}(u) \overset{a}{\mathcal{T}'_-} L^{\mathrm{t}_a}_{aN}(-u+\delta)R^{\mathrm{t}_a\mathrm{t}_b}_{ba}(u_-) \\
&= L_{bN}(v) \overset{b}{\mathcal{T}'_-} L_{aN}(u)R^{\mathrm{t}_b}_{ba}(-u_+ +\delta)L^{\mathrm{t}_b}_{bN}(-v+\delta) \overset{a}{\mathcal{T}'_-} L^{\mathrm{t}_a}_{aN}(-u+\delta)R^{\mathrm{t}_a\mathrm{t}_b}_{ba}(u_-) \\
&= L_{bN}(v)L_{aN}(u) \overset{b}{\mathcal{T}'_-} R^{\mathrm{t}_b}_{ba}(-u_+ +\delta) \overset{a}{\mathcal{T}'_-} R^{\mathrm{t}_a\mathrm{t}_b}_{ba}(u_-)L^{\mathrm{t}_a}_{aN}(-u+\delta)L^{\mathrm{t}_b}_{bN}(-v+\delta).
\end{aligned}$$

In other words, this equation is reduced to

$$R_{ab}(u_-) \overset{a}{\mathcal{T}'_-} R^{\mathrm{t}_a}_{ab}(-u_+ +\delta) \overset{b}{\mathcal{T}'_-} = \overset{b}{\mathcal{T}'_-} R^{\mathrm{t}_b}_{ba}(-u_+ +\delta) \overset{a}{\mathcal{T}'_-} R^{\mathrm{t}_a\mathrm{t}_b}_{ba}(u_-).$$

We proceed to do the above reduction repeatedly until all of the L operators beside the K_- matrix disappear. Finally, we obtain the reflection equation about K_- only. Now, the reflection equations of $K_\pm$ are

$$\begin{aligned}
&R^{\mathrm{t}_1\mathrm{t}_2}_{21}(-u_-) \overset{1}{K}{}^{\mathrm{t}_1}_+(u)R^{\mathrm{t}_2}_{21}(u_+ -\delta) \overset{2}{K}{}^{\mathrm{t}_2}_+(v) = \overset{2}{K}{}^{\mathrm{t}_2}_+(v)R^{\mathrm{t}_1}_{12}(u_+ -\delta) \overset{1}{K}{}^{\mathrm{t}_1}_+(u)R_{12}(-u_-) \\
&R_{12}(u_-) \overset{1}{K}_-(u)R^{\mathrm{t}_1}_{12}(-u_+ +\delta) \overset{2}{K}_-(v) = \overset{2}{K}_-(v)R^{\mathrm{t}_2}_{21}(-u_+ +\delta) \overset{1}{K}_-(u)R^{\mathrm{t}_1\mathrm{t}_2}_{21}(u_-)
\end{aligned} \tag{44}$$

and the transfer matrix $t(u)$ becomes

$$t(u) = \mathrm{tr}K_+(u)T(u)K_-(u)T^{\mathrm{t}}(-u+\delta). \tag{45}$$

We note that there is no obvious relation between the $K_\pm$ matrices. If some symmetry conditions are used, a relation between the K_+ and K_- matrices may be found.

For example, the R-matrix in [13] has PT symmetry and crossing unitarity

$$\begin{aligned}
R_{12}(u) &= R^{\mathrm{t}_1\mathrm{t}_2}_{21}(u) \\
R_{12}(u) &= \overset{1}{V} R^{\mathrm{t}_2}_{12}(-u-\rho) \overset{1}{V}{}^{-1}.
\end{aligned} \tag{46}$$

By PT symmetry, we find that there is an isomorphism between the boundary matrices:

$$K_-(u) = K^{\mathrm{t}}_+(-u+\delta). \tag{47}$$

If both PT symmetry and crossing unitarity are considered, there is another relation

$$K_-(u) = K^{-1}_+(u+\rho)M^{-1} \qquad M = V^{\mathrm{t}}V. \tag{48}$$

From (47) and (48), this implies

$$K^{\mathrm{t}}_+(-u+\delta)MK_+(u+\rho) = \mathrm{Id} \qquad K^{\mathrm{t}}_-(-u+\delta)K_-(u-\rho)M = \mathrm{Id}. \tag{49}$$

These equations may be regarded as constraint conditions on δ.

Now, it is interesting to compare our commutative quantities with those of Mezincescu and Nepomechie [13]. Using the conditions of (46) and the unitarity condition $R_{12}(u)R_{21}(-u) = \xi(u)$, we obtain

$$\begin{aligned} R_{12}^{t_1}(-u-\rho) &= (\overset{1}{V})^{t_1}\, R_{12}^{t_1t_2}(u)(\overset{1}{V}{}^{-1})^{t_1} \\ &= \xi(-u)(\overset{1}{V})^{t_1}\, R_{12}^{-1}(-u)(\overset{1}{V}{}^{-1})^{t_1} \end{aligned}$$

or

$$R_{12}^{t_1}(-u+\delta) = \xi(-u+\rho+\delta)(\overset{1}{V})^{t_1}\, R_{12}^{-1}(-u+\rho+\delta)(\overset{1}{V}{}^{-1})^{t_1}.$$

If $L_n(u)$ is defined as $L_n(u) \equiv L_{an}(u) = R_{an}(u)$, we obtain

$$\begin{aligned} L_n^{t}(-u+\delta) &= \xi(-u+\rho+\delta)V^{t}L_n^{-1}(-u+\rho+\delta)(V^{-1})^{t} \\ T^{t}(-u+\delta) &= L_1^{t}(-u+\delta)L_2^{t}(-u+\delta)\ldots L_N^{t}(-u+\delta) \\ &= \xi^N(-u+\rho+\delta)V^{t}T^{-1}(-u+\rho+\delta)(V^{-1})^{t}. \end{aligned}$$

In other words, the transfer matrix (45) becomes

$$\begin{aligned} t(u) &= \mathrm{tr}K_+(u)T(u)K_-(u)(\xi^N(-u+\rho+\delta)V^{t}T^{-1}(-u+\rho+\delta)(V^{-1})^{t}) \\ &= \xi^N(-u+\rho+\delta)\mathrm{tr}((V^{-1})^{t}K_+(u))T(u)(K_-(u)V^{t})T^{-1}(-u+\rho+\delta). \end{aligned} \tag{50}$$

It is convenient to multiply (50) by $\xi^{-N}(-u+\rho+\delta)$ before we regard it as the commutative quantities, i.e.

$$t(u) = \mathrm{tr}K'_+(u)T(u)K'_-(u)T^{-1}(-u+\rho+\delta).$$

If δ is equal to $-\rho$, it is just the same as that in [13].

As one of the main results of our paper, we have constructed the commutative quantities with the unitarity condition of the quantum R-matrix only. As discussed in the classical case, with symmetry conditions of (46), we can find another form of the commutative quantities, and these two forms are in fact the same when both are regarded as commutative quantities.

Finally, we study the classical counterparts of the reflection equations (44) by modifying the unitarity condition to $R_{12}(u)R_{21}(-u) = \mathrm{Id}$. In the classical limit, as $\hbar \to 0$, one has [3, 20]:

$$[\ ,\ ] = -\mathrm{i}\hbar\{\ ,\ \} \qquad R(u) = \mathrm{Id} + \mathrm{i}\hbar r(u) + o(\hbar^2).$$

So the unitarity condition implies $r_{12}(u) = -r_{21}(-u)$ and the quantum YBE goes over into the classical YBE. We find that reflection equations (44) just turn into equation (24) in which $r(\alpha, \beta)$ is now equal to $r(\alpha-\beta)$. In contrast with the quantum case, there is an isomorphism between K_+ and K_-, which is $K_+(\alpha) \to K_-^{-1}(\alpha)$ in the classical case.

6. Conclusion and discussion

In this paper, we have obtained three possible generating functions for the integrals of motion in classically integrable field theory on a finite interval with independent boundary conditions at each end. As constraint conditions, we find the algebra and evolution equations of $K_\pm$ matrices. In contrast with other methods, a new parameter is added onto the spectral parameter, and we expect it shall simplify the procedure of solving the $K_\pm$ matrices effectively. In ATFT, we prove that these generating functions are equivalent to each other and their links are also discussed. Our results show that two of these generating functions are always valid in both the real and imaginary coupling constant cases.

It is remarkable that no symmetry condition of the r-matrix is used when we regard quantities (11)–(13) as generating functions for the integrals of motion, so we expect it can be applied to more integrable models than are in [3, 5]. As demonstrated in section 4, the added parameter δ improves this possibility.

We have also extended our results to a quantum spin chain and have proved that the unitarity condition of the quantum R-matrix is sufficient to construct commutative quantities with boundary. The reflection equations of $K_\pm$ are obtained. The relation between the boundary $K_\pm$ matrices found when PT symmetry and the crossing unitarity condition of the R-matrix are considered. With these symmetry conditions, we have also found another form of the commutative quantities different from that defined in [13]. Finally, we have found that classical counterparts of the quantum reflection equations are just those which are obtained from the classical quantity (12).

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